

# THE ISOPERIMETRIC PROBLEM IN HIGHER CODIMENSION

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ABSTRACT. We consider three generalizations of the isoperimetric problem to higher codimension and provide results on equilibrium, stability, and minimization.

## 1. INTRODUCTION

The classical isoperimetric problem in an  $n$ -dimensional Riemannian manifold seeks an  $(n - 1)$ -dimensional surface  $S$  of least area bounding a region  $R$  of prescribed volume. To generalize the problem to  $m$ -dimensional surfaces  $S$  ( $1 \leq m \leq n - 2$ ) requires a notion of enclosed volume. We present three alternatives:

- (1) infimum  $v(S)$  of volumes of  $(m + 1)$ -dimensional surfaces bounded by  $S$ ,
- (2)  $\omega$ -volume  $\int_S \omega$  for some given smooth  $m$ -form  $\omega$ ,
- (3) in  $\mathbf{R}^n$  multi-volume, i.e., volume enclosed by projection to each axis  $(m + 1)$ -dimensional vector subspace of  $\mathbf{R}^n$ , or equivalently prescribed  $\omega$ -volume for all  $m$ -forms  $\omega$  with  $d\omega$  constant.

For the first notion, perhaps the most natural, Almgren ([3], 1986) proved that in  $\mathbf{R}^n$ , round spheres are uniquely isoperimetric.

The second notion was introduced by Salavessa ([24], 2010), actually in terms of  $\Omega = d\omega$ ; note that for any surface  $R$  bounded by  $S$ ,

$$\int_R \Omega = \int_S \omega.$$

Given an exact form  $\Omega$ ,  $\omega$  is defined up to a closed form. If  $\Omega$  is a constant  $(m + 1)$ -form in  $\mathbf{R}^n$ , it follows from Almgren's result that round spheres are uniquely isoperimetric. Salavessa [24] proves that round spheres uniquely satisfy some strong stability hypotheses.

The third notion was introduced by Morgan ([21], 2000), who characterized isoperimetric curves (not necessarily round) and gave examples of non-round isoperimetric surfaces.

We could more generally consider surfaces  $S$  with prescribed boundary as well as prescribed volume. In case (1), volume must then be measured with respect to a given reference surface with the given boundary other than itself. In the closely related (higher dimensional)

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thread problem (see [12], [[7], Chap. 10], [23, 19]), which makes sense only for given reference surface, one fixes an area smaller than the reference surface and minimizes volume. The fact that one is minimizing rather than maximizing volume makes it easy to prove that the thread has constant mean curvature inside the volume-minimizing surface where it is smooth; otherwise one could move the surface inward where the curvature is small and outward less where it is larger, preserving area but reducing volume, because volume is at most the volume of the perturbed surface. A further difficulty for our case of prescribed volume is that there is no obvious perturbation preserving volume. The thread problem is roughly equivalent to minimizing area for a prescribed upper bound on volume. Least area is a continuous function of prescribed volume, but unless it is decreasing-increasing, minimum and maximum volume are not continuous functions of prescribed area. Figure 1 suggests possible relationships between area and volume, although we do not know a specific example that exhibits all these possibilities.

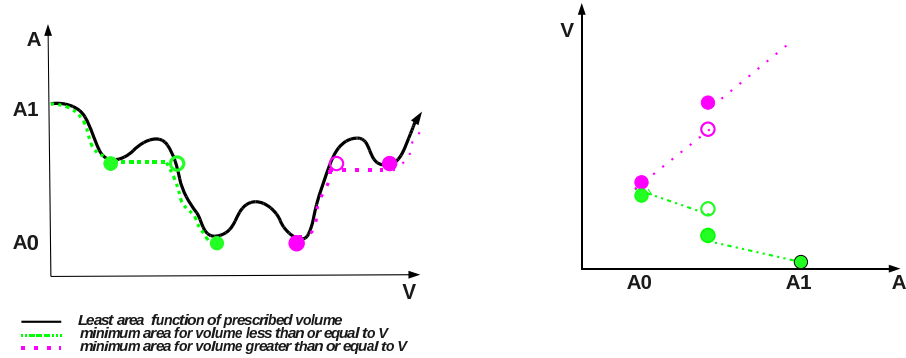


FIGURE 1. Least area (in black) is a continuous function of prescribed volume, but minimum and maximum volume are not continuous functions of prescribed area. In green is minimum area for volume less than or equal to  $V$ . In purple is minimum area for volume greater than or equal to  $V$ .

In case (1), we could also work in the larger context of unoriented surfaces.

To allow our surfaces to have singularities, we work in the context of the locally integral currents of geometric measure theory [[18], Chaps. 4 and 9].

In earlier work R. Gulliver [14, 15], F. Duzaar and M. Fuchs ([8, 10], and especially [[9], Thm. 3.2]), and Duzaar and K. Steffen [11], seek surfaces with prescribed mean curvature vector by minimizing  $A - \lambda V$ . Gulliver [[14], p. 118] gives one interpretation of a helical minimizer as the path of “a charged particle moving in a magnetic field.”

This paper provides a unified treatment on minimizing area for the three notions (1)-(3) of prescribed volume. Section 2 discusses equilibrium conditions. Section 3 discusses existence and regularity. Section 4 considers the question of whether round spheres are the only isoperimetric or stable surfaces. We conjecture that in  $\mathbf{R}^n$ , round  $m$ -spheres  $S_0$  are

the only smooth stable surfaces  $S$  for given volume  $v(S)$  or given  $\Omega$ -volume for constant  $\Omega$  (although not for given multi-volume).

## 2. STATIONARY SURFACES

This section presents the equilibrium conditions for the isoperimetric problem for the three types of volume constraints, generalizing the equilibrium condition of constant mean curvature of codimension 1. Higher codimension presents new issues of smoothness and degeneracy.

We will consider perturbations  $S_t$  of an  $m$ -dimensional surface  $S = S_0$  under nice smooth families  $F_t$  ( $0 \leq t < t_1$ ) of diffeomorphisms of  $M$ , with  $F_0$  the identity. More specifically, we will assume that  $F_t$  is  $C^3$  with spatial derivatives of orders 2 and 3 bounded, which includes scaling in  $\mathbf{R}^n$ . If  $S$  has infinite area, we will assume that the  $F_t$  equal the identity outside a fixed compact set.

**Definition 2.1.** *We call  $S$  stationary if the (one-sided) first derivative of the area  $A = |S|$  of  $S$  is nonnegative whenever  $S_t$  respect the volume constraint. We call  $S$  stable if small perturbations respecting the volume constraint have no less area.*

If  $S$  is stationary, then the second variation depends only on the initial variation vectorfield  $\mathbf{v} = \partial F / \partial t$ .

In the classical case of codimension 1 ( $m = n - 1$ ), a stationary surface has *generalized mean curvature*  $\mathbf{H}$  (defined almost everywhere) of constant magnitude and normal to the surface, i.e., for every smooth variation vectorfield  $\mathbf{v}$ , initially

$$dA/dt = - \int_S (n-1) \mathbf{H} \cdot \mathbf{v}.$$

This case is easy because volume  $V$  also varies smoothly and non-degenerately; initially

$$dV/dt = - \int_S \mathbf{n} \cdot \mathbf{v},$$

where  $\mathbf{n}$  is the inward unit normal, defined almost everywhere. In higher codimension  $V$  need not vary smoothly, as when  $S$  bounds multiple volume-minimizing surfaces. Nor need  $V$  vary non-degenerately: smooth families  $F_t$  with  $dV/dt$  initially 0 sometimes cannot be modified to keep  $V$  constant (see §4.2 and §4.3). The generalized mean curvature vector exists as long as  $dA/dt$  is a bounded operator; see Allard [1] for details in a general (“varifold”) setting.

The rest of this section attempts to recover a constant-magnitude generalized mean curvature vector for stationary surfaces in higher codimension for the three definitions of prescribed volume. In the most difficult first case of prescribed volume  $v(S)$ , Proposition 2.2 requires a strong smoothness hypothesis, while the more useful Proposition 2.3 uses a stronger notion of *stationary*. In the other two cases of prescribed  $\omega$ -volume and prescribed multi-volume, volume varies smoothly but degeneracy can be an issue.

**Proposition 2.2.** *Let  $S$  be a boundary in a smooth Riemannian manifold  $M$ . Suppose that*

- (1) *there is a nonzero measurable vectorfield  $\mathbf{G}$  on  $S$  such that for any smooth variation vectorfield  $\mathbf{v}$ , the volume  $V = v(S)$  is smooth and initially  $dV/dt = - \int_S \mathbf{G} \cdot \mathbf{v}$ .*

*Then  $S$  is stationary if and only if the generalized mean curvature vector  $\mathbf{H}$  is a constant times  $\mathbf{G}$ .*

*Remarks.* If  $S$  bounds a volume-minimizing surface  $R$ , then any such  $\mathbf{G}$  is weakly the inward unit conormal, i.e.,  $d|R|/dt$  is initially  $-\int_S \mathbf{G} \cdot \mathbf{v}$ . If two such volume-minimizing surfaces or sufficiently regular minimal surfaces are indecomposable and smooth submanifolds with boundary at some point of  $S$ , they are equal, as follows by partial differential equations [[20], Sect. 7], using for continuation the indecomposability of  $S$  and the fact that volume-minimizing hypersurfaces are regular except possibly for a codimension-2 singular set [4]. In particular, if  $S$  is a smooth, connected submanifold of  $\mathbf{R}^n$ , then the volume-minimizing surface is unique because volume-minimizing surfaces are regular at extreme points of  $S$  by Allard's boundary regularity theorem [2]. Conversely,

(2) *we conjecture that hypothesis (1) holds whenever  $S$  bounds a unique volume-minimizing surface.*

On the other hand, (1) clearly fails if  $S$  bounds two different volume-minimizing surfaces with distinct conormals. For an extreme negative example, let  $M$  be the round 2-sphere and let  $S$  be two antipodal points.

By work of B. White [28], (2) holds in  $\mathbf{R}^n$  and in compact real-analytic  $n$ -dimensional ambients  $M$ , as long as all volume-minimizing surfaces are smoothly immersed manifolds with boundary, which in turn holds if  $S$  has dimension  $n - 2 \leq 5$  or if  $S$  has dimension 1 and we admit unoriented volume-minimizing surfaces ([18], Chap. 8), [2]; unoriented 2-dimensional area-minimizing surfaces have no branch points or singularities except where sheets cross orthogonally). Furthermore almost every  $S$  bounds a unique volume-minimizing surface ([20], Thm. 7.1 and Rmk.] with [4]).

Apparently many smooth surfaces with (even parallel) mean curvature vector of constant length are not stationary for prescribed volume, such as smooth minimal submanifolds  $S$  of the unit sphere in  $\mathbf{R}^n$  (at least for  $n \leq 7$ ) bounding unique volume-minimizing surfaces other than the cone, such as  $\mathbf{S}^1(1/\sqrt{5}) \times \mathbf{S}^2(2/\sqrt{5})$  in  $\mathbf{R}^5$ , if indeed that bounds a unique volume-minimizing surface. (If  $S$  were stationary, the volume-minimizing surface, smooth along  $S$  by Allard's boundary regularity theorem, would by Proposition 2.2 have radially inward conormal and therefore equal the cone by the PDE argument described earlier in these remarks.) Conversely, we doubt that all stationary surfaces have parallel mean curvature and give a probable counterexample in a manifold in the Remarks after Proposition 2.3. If one allows prescribed boundary as well as prescribed volume, Yau's characterization of 2-dimensional surfaces with parallel mean curvature (Sect. 4.4) implies that some 2-dimensional isoperimetric surfaces in  $\mathbf{R}^4$  have nonparallel mean curvature, namely when the boundary is not contained in some  $\mathbf{S}^3$  or  $\mathbf{R}^3$  and the surface is non-minimal (as it must be for large prescribed volume).

One example where (1) holds is a round  $m$ -sphere  $S$  in  $\mathbf{R}^n$ , with  $\mathbf{G}$  the inward unit conormal to the flat ball and  $\mathbf{H}$  proportional to  $\mathbf{G}$ ; here an easy lower bound on volume is provided by the projection into the  $(m+1)$ -plane containing  $S$ . Conjecture (2) would imply that one nonround example is  $S = \mathbf{S}^3 \times \mathbf{S}^3$  in  $\mathbf{R}^8$ , with  $\mathbf{G}$  the inward unit conormal to the cone over  $S$ , which is famously volume-minimizing, but we don't see how to obtain the requisite lower bound on volume. Proposition 2.4 provides an alternative proof that  $S$  is stationary.

*Proof of Proposition 2.2.* If  $S$  bounds a volume-minimizing surface  $R$ , since  $dV \leq d|R|$  for  $d|R|$  positive or negative, by smoothness  $dV = d|R|$ ,  $d|R|/dt = dV/dt = -\int_S \mathbf{G} \cdot \mathbf{v}$ , and  $|\mathbf{G}| \leq 1$ .

Suppose that the generalized mean curvature vector  $\mathbf{H}$  exists and is a constant times  $\mathbf{G}$ . Then if  $V$  is constant, initially  $dV/dt = -\int_S \mathbf{G} \cdot \mathbf{v} = 0$ , so  $dA/dt = -\int_S \mathbf{H} \cdot \mathbf{v} = 0$ , i.e.,  $S$  is stationary.

Conversely, suppose that  $S$  is stationary. Since  $\mathbf{G}$  is nonzero, we can choose variation vectorfields  $\mathbf{v}_1$  and  $\mathbf{v}_2$  with disjoint supports such that initially  $dV/dt$  is nonzero for each of them. Given a point  $p$  of  $S$ , consider a neighborhood of  $p$  disjoint from the support of  $\mathbf{v}_1$  or  $\mathbf{v}_2$ , say  $\mathbf{v}_1$ , and let  $\mathbf{v}$  be a smooth variation vectorfield supported in that neighborhood. Now some linear combination  $\mathbf{w} = \mathbf{v} + c_1 \mathbf{v}_1$  has  $dV/dt = 0$  and hence by smoothness comes from a two-sided  $(-t_1 < t < t_1)$  volume-preserving family of diffeomorphisms of the form

$$F_t(x) = \exp_x(t\mathbf{v} + \varphi(t)c_1\mathbf{v}_1),$$

with  $\varphi(0) = 0$  and  $\varphi'(0) = 1$ . Since  $S$  is stationary, initially  $dA/dt = 0$ . Consequently if  $dV/dt$  for  $\mathbf{v}$  is nonzero, then  $dA/dV$  for  $\mathbf{v}$  is the same as it is for  $\mathbf{v}_1$ . In particular,  $dA/dV$  for  $\mathbf{v}_2$  is the same as it is for  $\mathbf{v}_1$ . Therefore  $dA/dV$  is a constant  $c$ . It follows that  $c\mathbf{G}$  is a generalized mean curvature vector for  $S$ .

**Proposition 2.3.** *Let  $S$  be a boundary in  $\mathbf{R}^n$  with finite area and volume  $v(S)$ . Let  $H_0 = |S|/(m+1)v(S)$ . If  $dA/dt \geq 0$  under smooth families of diffeomorphisms  $F_t$  for which  $\Delta v(S) \geq 0$ , then  $S$  has generalized mean curvature  $\mathbf{H}$  of magnitude bounded by  $H_0$ . Conversely if  $S$  is smooth with mean curvature vector  $\mathbf{H}$  a constant nonnegative multiple of the inward unit conormal of a volume-minimizing surface, then  $dA/dt \geq 0$  under smooth families of diffeomorphisms  $F_t$  for which  $\Delta v(S) \geq 0$ , and  $|\mathbf{H}| = H_0$ .*

*Proof.* The technical difficulty is that although the area of  $S = S_0$  varies smoothly under smooth perturbations  $S_t$ , the volume  $v(S_t)$  may not. Nevertheless for any smooth variation vectorfield  $\mathbf{v}$ , the change in  $V(t) = v(S_t)$  satisfies

$$(1) \quad \Delta V \geq -|\Delta t| \int_S |\mathbf{v}| - o(\Delta t)$$

because if there were surfaces bounded by  $S_t$  of smaller volume, adding on the volume swept out by the  $S_t$  would yield a surface bounded by  $S_0$  of less volume than  $V(0)$ , a contradiction. Given  $\varepsilon > 0$ , consider rescalings by a factor  $1 + at$  with  $a$  chosen such that initially

$$dV/dt = a(m+1)V(0) = \int_S |\mathbf{v}| + \varepsilon$$

and hence

$$dA/dt = a mA(0) = mH_0 \left( \int_S |\mathbf{v}| + \varepsilon \right).$$

After combining the original family with such rescalings, (1) becomes  $\Delta V \geq \varepsilon \Delta t - o(\Delta t) \geq 0$  for  $0 \leq t < t_1$  and hence by hypothesis  $dA/dt \geq 0$ . Therefore for the original family

$$dA/dt \geq -mH_0 \left( \int_S |\mathbf{v}| + \varepsilon \right);$$

since this holds for all  $\varepsilon > 0$ ,  $dA/dt \geq -mH_0 \int_S |\mathbf{v}|$ . Since this holds also for  $-\mathbf{v}$ ,  $|dA/dt| \leq mH_0$ , and this holds for every smooth variation  $\mathbf{v}$ . It follows that there is a generalized curvature vector  $\mathbf{H}$  of magnitude bounded by  $H_0$ .

Conversely, suppose that  $S$  is smooth with mean curvature vector  $\mathbf{H}$  a constant nonnegative multiple of the inward unit conormal  $\mathbf{n}$  of a volume-minimizing surface  $R$ . For any smooth family of diffeomorphisms for which  $\Delta v(S) \geq 0$ ,

$$0 \leq d|R|/dt = - \int_S \mathbf{n} \cdot \mathbf{v}.$$

Since  $\mathbf{H}$  is a constant positive multiple of  $\mathbf{n}$ ,  $dA/dt = - \int_S m\mathbf{H} \cdot \mathbf{v} \geq 0$ .

Under scaling,  $dA/dt = - \int_S m\mathbf{H} \cdot \mathbf{v}$  and  $dV/dt = - \int_S \mathbf{n} \cdot \mathbf{v}$ . Also for scaling  $A^{1/m} = cV^{1/(m+1)}$  and  $dA/dV = mA/(m+1)V = mH_0$ . Therefore  $\mathbf{H} = H_0\mathbf{n}$ .

*Remarks.* The first statement of the converse and its proof hold in any smooth Riemannian manifold. The hypothesis, when  $R$  is not smooth along  $S$ , need only hold weakly:  $d|R|/dt$  proportional to  $- \int_S \mathbf{H} \cdot \mathbf{v}$ .

In  $\mathbf{R}^2 \times [0, \varepsilon]$  with the top and bottom identified with an appropriate slight twist, the helix is stationary (by the converse, assuming that the helicoid is area minimizing) and probably isoperimetric.

**Proposition 2.4.** *For  $p \geq 1$ ,  $\mathbf{S}^p \times \mathbf{S}^p$  in  $\mathbf{R}^{2p+2}$  is strongly stationary as in Proposition 2.3.*

*Proof.* For  $p \geq 3$  the cone is famously volume minimizing ([6], see [[18], §10.7]), and the result follows immediately from Proposition 2.3. For the general case let  $R_1$  be a volume-minimizing surface bounded by  $S$ . By Allard's boundary regularity theorem [2],  $R_1$  is a smooth submanifold with boundary along  $S$ . Let  $R_2$  be its image under the symmetry switching the first two coordinates with the last two coordinates. Then  $\mathbf{H}$ , which is in the unique symmetric normal direction to  $\mathbf{S}^p \times \mathbf{S}^p$ , must be proportional to the sum  $\mathbf{n}_1 + \mathbf{n}_2$  of the conormals. For any smooth family of diffeomorphisms for which  $\Delta v(S) \geq 0$ ,

$$0 \leq d|R_i|/dt = - \int_S \mathbf{n}_i \cdot \mathbf{v}.$$

Since  $\mathbf{H}$  is a proportional to  $\mathbf{n}_1 + \mathbf{n}_2$ ,  $dA/dt = - \int_S m\mathbf{H} \cdot \mathbf{v} \geq 0$ .

We now consider the second case of prescribed  $\omega$ -volume. Salavessa [[24], Thm. 2.1] proves the following equilibrium condition in the narrower context where  $d\omega \lrcorner T$  has constant length.

**Proposition 2.5** (cf. [24], Thm. 2.1). *Consider a cycle (surface without boundary)  $S$  with unit tangent  $m$ -vector  $T$  in a smooth Riemannian manifold  $M$  with smooth  $m$ -form  $\omega$ .  $S$  is stationary for positive prescribed  $\omega$ -volume if the mean curvature vector  $\mathbf{H}$  is proportional to  $d\omega \lrcorner T$  (weakly). Further suppose that  $d\omega \lrcorner T$  is not identically 0 or that  $M = \mathbf{R}^n$  and  $d\omega$  is constant. If  $S$  is stationary, then the mean curvature vector  $\mathbf{H}$  is proportional to  $d\omega \lrcorner T$  (weakly).*

*If  $M = \mathbf{R}^n$ ,  $d\omega$  is constant and simple, and  $S$  is stationary, smooth, connected, and bounded, then  $S$  lies in an associated  $(m+1)$ -plane and is round.*

That the mean curvature vector  $\mathbf{H}$  is proportional to  $d\omega \lrcorner T$  (weakly) means that there is a constant  $c$  such that for any smooth variation vectorfield  $\mathbf{v}$ , initially

$$dA/dt = c \int_S (d\omega \lrcorner T)(\mathbf{v}) = c \int_S d\omega(T \wedge \mathbf{v}).$$

(The generalized mean curvature  $\mathbf{H}$  is characterized by  $dA/dt = - \int_S m \mathbf{H} \cdot \mathbf{v}$  for any smooth variation vectorfield  $\mathbf{v}$ , and one often identifies the vector  $\mathbf{H}$  with the 1-form  $\mathbf{H} \cdot$ .)

*Remark.* The additional hypothesis for the converse is necessary. For example, let  $S$  be any embedding of the hypersphere of finite area in  $\mathbf{R}^n$ , with inside  $U$  and outside  $V$ . Let  $f, g$  be nonnegative  $C^\infty$  functions with support  $U \cup S$  and  $V \cup S$  respectively, and let  $\Omega = (f - g)dx_1 \wedge \dots \wedge dx_n$ . Then  $S$  is isoperimetric for prescribed  $\Omega$ -volume; indeed it is the only surface with its  $\Omega$ -volume.

A similar hypothesis appears for example in [[9], Thm. 5.1].

*Proof of Proposition 2.5.* For every smooth variation vectorfield  $\mathbf{v}$  on  $S$ , initially

$$dV/dt = \int_S (d\omega \lrcorner T)(\mathbf{v}),$$

basically because by Stokes's theorem the change in volume is the integral of  $d\omega$  over the volume swept out. It follows immediately that if the mean curvature vector  $\mathbf{H}$  is proportional to  $d\omega \lrcorner T$  (weakly), i.e. if

$$dA/dt = \lambda \int_S (d\omega \lrcorner T)(\mathbf{v}),$$

then  $V$  constant implies that  $dA/dt = 0$ , so  $S$  is stationary.

Conversely, suppose that  $S$  is stationary and  $d\omega \lrcorner T$  is not identically 0. Then the constraint is nonsingular as well as smooth, so for some Lagrange multiplier  $\lambda$ ,  $dA/dt = \lambda (dV/dt)$ , as desired. Alternatively suppose that  $S$  is stationary, that  $M = \mathbf{R}^n$ , that  $d\omega$  is constant, and that  $d\omega \lrcorner T$  is 0 almost everywhere. Then variations of the form  $S_t = S + t\mathbf{v}$  with  $\mathbf{v}$  of the special form  $\mathbf{v} = \varphi \cdot \mathbf{v}_0$  for some smooth scalar function  $\varphi$  and fixed vector  $\mathbf{v}_0$  preserve  $\omega$ -volume. Since  $S$  is stationary,  $dA/dt$  is initially 0. Since such vectorfields  $\mathbf{v}$  span the space of all smooth variation vectorfields,  $\mathbf{H}$  is 0.

Finally suppose that  $M = \mathbf{R}^n$ ,  $d\omega$  is constant and simple, and  $S$  is stationary, smooth, connected, and bounded. We may assume that  $d\omega$  is  $dx_1 \wedge \dots \wedge dx_{m+1}$ . For coordinates  $(x, y)$  on  $\mathbf{R}^{m+1} \times \mathbf{R}^{n-m-1}$ , consider a family of diffeomorphisms given on  $S$  by  $F_t(x, y) = (x, y/(1+t))$ . Since they preserve volume,  $dA/dt$  initially must be 0, which means that  $S$  is everywhere horizontal and lies in a horizontal copy of  $\mathbf{R}^{m+1}$ . Since  $S$  has constant mean curvature (nonzero because  $S$  is bounded),  $S$  is round by Alexandrov's Theorem.

Finally we consider the third case of prescribed multi-volume.

**Proposition 2.6** ([21], Thm. 2.2). *A boundary  $S$  with unit tangent  $m$ -vector  $T$  in  $\mathbf{R}^n$  is stationary for prescribed multi-volume if and only if for some constant  $(m+1)$ -form  $\Omega$ , the mean curvature  $\mathbf{H}$  of  $S$  weakly satisfies  $m\mathbf{H} \cdot = \Omega \lrcorner T$ , i.e., for any smooth variation vectorfield  $\mathbf{v}$ , initially*

$$(2) \quad dA/dt = - \int_S m \mathbf{H} \cdot \mathbf{v} = - \int_S (\Omega \lrcorner T)(\mathbf{v}) = - \int_S \Omega(T \wedge \mathbf{v}).$$

*Proof.* Assume that (2) holds. Choose a smooth  $\omega$  such that  $\Omega = d\omega$ . By Proposition 2.5,  $S$  is stationary for prescribed  $\omega$ -volume, and hence for prescribed multi-volume.

Conversely, assume that  $S$  is stationary. For any covector  $\Omega = d\omega$ , consider the associated volume  $\int_S \omega$ . (Fixing multi-volume is equivalent to fixing all such volumes or just the axis volumes  $V_I$ .) For every smooth variation vectorfield  $\mathbf{v}$  on  $S$ , initially

$$dV_I/dt = \int_S (dx_I \lrcorner T)(\mathbf{v}).$$

We consider variations of the forms  $S_t = S + t\mathbf{v}$  with  $\mathbf{v}$  of the special form  $\mathbf{v} = \varphi \cdot \mathbf{v}_0$  for some smooth scalar function  $\varphi$  and fixed vector  $\mathbf{v}_0$ , which span the space of all smooth variations. A variation of this simple form never alters volumes for  $\Omega$  outside

$$\text{span}\{T \wedge \mathbf{v}_0 : \mathbf{v}_0 \in \mathbf{R}^n, \text{ values of } T \text{ at Lebesgue points}\} \subset \wedge^{m+1} \mathbf{R}^n.$$

The constraint for such volumes is nonsingular as well as smooth, as can be seen by consideration of variations supported in small neighborhoods of Lebesgue points of  $T$ . Therefore for some Lagrange multiplier  $\lambda = (\lambda_I)$ ,  $dA/dt = \sum \lambda_I (dV_I/dt)$ , so with  $\Omega = -\sum \lambda_I dx_I$

$$dA/dt = - \int_S (\Omega \lrcorner T)(\mathbf{v}),$$

as desired.

### 3. EXISTENCE AND REGULARITY OF ISOPERIMETRIC SURFACES

This section presents standard geometric measure theory results on existence and regularity.

**3.1. Existence.** *If  $M$  is compact or if  $M = \mathbf{R}^n$  and  $d\omega$  is constant, then isoperimetric surfaces  $S$  exist for all prescribed volumes  $v(S)$ ,  $\omega$ -volumes, and multi-volumes and are compact.*

*Proof.* If  $M$  is compact there are no issues, one just takes a minimizing sequence and applies the Compactness Theorem [[18], Chap. 5] to get a solution in the limit. In  $\mathbf{R}^n$ , local compactness still provides a possibly unbounded area-minimizing limit among locally integral currents [[18], §9.1]. By Propositions 2.3, 2.5, 2.6, an isoperimetric surface has constant-magnitude mean curvature. Thence “monotonicity” [[1], 5.1(3)] yields a positive lower bound on the area inside a unit ball about every point of  $S$ , and it follows that  $S$  is compact.

A more serious problem is that there may be volume loss to infinity. One uses a concentration lemma and translation to obtain a minimizer with nonzero volume [[18], §13.4]. Then for prescribed volume  $v(S)$  or prescribed  $\omega$ -volume, one uses scaling (and a flip of orientation if necessary) to obtain the prescribed volume. For prescribed multi-volume one repeats the process countably many times to recover all the volume [[18], §13.4].

**3.2. Regularity.** By Allard’s regularity theorem [[1], Sect. 8], any surface with weakly bounded mean curvature is a  $C^{1,\alpha}$  submanifold on an open dense set. By Propositions 2.3, 2.5, 2.6, this includes all three types of isoperimetric surfaces in  $\mathbf{R}^n$ , assuming  $d\omega$  constant. It probably includes isoperimetric surfaces for prescribed volume  $v(S)$  in smooth Riemannian manifolds, but we do not know how to prove that.

For a negative example for prescribed  $\omega$ -volume in  $\mathbf{R}^n$  with  $d\omega$  nonconstant, see the Remark after Proposition 2.5.



It is not known whether isoperimetric surfaces for prescribed volume  $v(S)$  in smooth Riemannian manifolds and for prescribed multivolume in  $\mathbf{R}^n$  enjoy the same regularity as area-minimizing surfaces without volume constraints, even for the easier Lagrange multiplier problem; cf. [[18], Chap. 8], [[8], §5], [[11], Intro. and 5.5(iii)]. In general, it is not known even whether a tangent cone is minimizing, because the cost of small volume adjustments is not known to be linear. (Note e.g. the extra hypothesis required in [[9], Thm. 5.1].)

#### 4. ROUND SPHERES UNIQUELY MINIMIZING OR STABLE?

This section proves that round spheres are uniquely minimizing for all three volume constraints and conjectures that they are uniquely stable in  $\mathbf{R}^n$  for prescribed volume  $v(S)$  and for prescribed  $\Omega$ -volume for  $\Omega$  constant (but not for prescribed multi-volume).

**Proposition 4.1.** *Round  $m$ -spheres  $S_0$  are uniquely minimizing for all three cases; for prescribed  $\omega$ -volume (case 2) we need to assume  $\Omega = d\omega$  constant and maximum on the  $(m+1)$ -ball bounded by  $S_0$ .*

*Proof.* Case (1), prescribed volume  $v(S)$ . Almgren [3], indeed mod  $v$  for all  $v$ .

Case (2), prescribed  $\omega$ -volume. We may assume  $\Omega = d\omega$  is 1 on the disc  $D$ . Now let  $S$  be any surface with the same  $\omega$ -volume, and let  $R$  be a volume-minimizing surface bounded by  $S$ . Then

$$|D| = \int_D \Omega = \int_R \Omega \leq |R|.$$

By Case (1),  $|S_0| \leq |S|$ , with equality only if  $S$  is a round sphere and  $\Omega = 1$  on  $R$ .

Case (3), prescribed multi-volume, follows from Case (2) with  $\Omega$  the simple form dual to the disc.

*Remark.* Salavessa [25] proves the weaker result that associated round spheres have non-negative second variation for prescribed  $\Omega$ -volume for  $\Omega$  the Kähler form on  $\mathbf{R}^6$ .

The following conjecture would generalize a codimension-1 stability theorem of Barbosa and do Carmo [5] to higher codimension.

**4.2. Conjecture.** *In  $\mathbf{R}^n$ , round  $m$ -spheres  $S_0$  are the only smooth stable surfaces  $S$  for given volume  $v(S)$  or given  $\Omega$ -volume for constant  $\Omega$  (although not for given multi-volume [[21], Cor. 3.2]).*

*Proof for  $\Omega$ -volume for  $m = 1$ .* By [[21], Thm. 3.1], which applies to stationary as well as minimizing curves, a stationary closed curve for prescribed multi-volume or equivalently for prescribed  $\Omega$ -volume is of the form

$$C(s) = a_0 + a_1 e^{i w_1 s} e_1 + \dots + a_k e^{i w_k s} e_{2k-1},$$

with  $w_j$  increasing positive integers. If  $k = 1$ , this curve is a circle. If  $k > 1$ , this curve is neither minimizing nor stable for given  $\Omega$ -volume: in the second component, which encircles the origin twice, enlarging one loop and shrinking the other reduces length to second order for fixed area. (This variation does not preserve multi-volume because it alters area

in the  $e_{14}$  plane for example. Indeed, this curve is minimizing for prescribed multi-volume [[21], Cor. 3.2].)

*Remarks.* The proof that round spheres are the only minimizers dates from 1986 [Almgren [3]]. Generalizing the codimension 1 proof of Barbosa-do Carmo [5] and Wente [27] seems to need mean curvature parallel on  $S$  (see §4.3 below). Otherwise second variation in the normal direction is more positive ([26], p. 171] or because  $dH/dt$  is less negative). It also seems to need normal bundle geometrically trivial. Salavessa [24] seems further to need a Minkowski-type hypothesis: in her completely different terminology “ $\int_M S(2 + h\|H\|)dM \leq 0$ .” If the scalar mean curvature  $H$  is not constant, one could replace  $H$  with its average except that the average value of the square of  $H$  is greater than the square of the average value. For prescribed volume  $v(S)$ , this conjecture remains open even for curves ( $m = 1$ ) in  $\mathbf{R}^3$ ; perhaps the circle is even the only isoperimetric-stationary one-component curve in  $\mathbf{R}^n$ .

Suppose that  $C$  is a counterexample for  $m = 1$  in  $\mathbf{R}^3$  for say volume  $\pi$ . Further suppose that  $C$  has as expected (see 2.2 and 2.3) curvature  $\kappa$  of magnitude  $|C|/2\pi > 1$  in the direction of the inward normal. By the isoperimetric inequality (or by Bol-Fiala for a disc),  $|C| > 2\pi$ . Alternatively, since  $\kappa = |C|/2\pi$ ,  $|C| = 2\pi\kappa$ , Gauss-Bonnet yields  $C^2/2\pi = \int |\kappa| > 2\pi\chi$ , again yielding  $|C| > 2\pi$  for a disc, since the Euler characteristic  $\chi \leq 1$  for cases of interest ( $R$  connected). Moreover, since the curvature of  $R$  along  $C$  vanishes, so does the curvature of  $R$  normal to  $C$ : locally as a graph  $f_{xx}$  and  $f_{yy}$  vanish, but not necessarily  $f_{xy}$ , so we don’t see how to prove e.g. that  $f_{yyy}$  vanishes and that  $R$  contains rays from the boundary and must be a flat disk.

**4.3. Second Variation.** The formula for the second variation, that is, the second derivative of area for a smooth family of perturbations, is given by Schoen [[26], p. 171]. Note that every variation vectorfield for a compact surface  $S$  for prescribed constant  $\Omega$ -volume in  $\mathbf{R}^n$  with  $dV/dt$  initially 0 is part of a 1-parameter family with fixed volume obtained by adjusting any 1-parameter family with  $dV/dt$  initially 0 by continuous rescalings by homotheties. This generalizes to exact nonconstant  $\Omega$ -volume in manifolds as long as there is a variation vectorfield (like the one for scaling) for which  $dV/dt$  is not zero. This corresponds to the fact that if  $f$  is a smooth function on  $\mathbf{R}^n$ ,  $\partial f/\partial x_1 = 0$ , and  $\text{grad } f \neq 0$ , then  $f$  vanishes on a smooth horizontal curve through 0.

On the other hand, even minimizers for prescribed multi-volume can be unstable for variations which preserve multi-volume to first order (and hence cannot correspond to 1-parameter multi-volume-preserving families). For example, consider the curve  $(e^{is}, e^{2is})$  in  $\mathbf{R}^2 \times \mathbf{R}^2$ , which is isoperimetric [[21], Cor. 3.2]. In the second factor the curve is two copies of the unit circle. Shrinking one and expanding the other preserves multi-volume to first order but reduces length to second order.

Conjecture 2.2(2) would imply that this curve is not even stationary for prescribed volume  $v(S)$ . The unique volume-minimizing surface bounded by this curve is  $\{w = z^2\}$ , because complex analytic varieties are uniquely volume minimizing [[18], 6.3]. Note that  $\mathbf{H}$  is proportional to  $-(e^{is}, 4e^{2is})$ , while the inward conormal is proportional to  $-(dz, 2zdz)$  hence to  $-(e^{is}, 2e^{2is})$ . By Proposition 2.2 and Remarks, the curve is not stationary.

Incidentally, this curve is not a graph over every axis plane: its projection in the  $x_1x_4$ -plane is a figure 8 enclosing signed area 0. In this case the problem of prescribing *unsigned* areas would have a different solution, presumably circles in the  $x_1x_2$ -plane and the  $x_3x_4$ -plane.

**4.4. Yau [29] on parallel mean curvature vector.** Yau [29] proved that every smooth 2-dimensional surface in  $\mathbf{R}^n$  with parallel mean curvature vector is one of four types:

- (1) constant-mean-curvature hypersurfaces in some  $\mathbf{R}^3 \subset \mathbf{R}^n$ ,
- (2) constant-mean-curvature hypersurfaces of some  $\mathbf{S}^3 \subset \mathbf{R}^n$ ,
- (3) minimal submanifolds of some hypersphere  $\mathbf{S}^{n-1} \subset \mathbf{R}^n$ ,
- (4) minimal submanifolds of  $\mathbf{R}^n$ .

One could study Lawson [17] and other examples of minimal surfaces in  $\mathbf{S}^3 \subset \mathbf{R}^4$  (type (2) and (3)).

*Example.*  $S = \mathbf{S}^1 \times \mathbf{S}^1$  in  $\mathbf{R}^4$ .  $\mathbf{H}$  is parallel (Yau type (2) and (3)), but  $S$  is not stationary even for fixed multi-volume, because it and all of its scalings have multi-volume 0. In particular there is no constant 3-form  $\Omega$  dual to  $T \wedge \mathbf{H}$  on  $S$ , a completely trivial consequence, since codimension-1 forms are simple and  $T \wedge \mathbf{H}$  is not constant. An obvious variable calibration candidate,  $dr \, r_1 d\theta_1 \, r_2 d\theta_2$  is not closed. We think that for general reasons there is a smooth classical calibration  $\Omega$  of a small band of the cone over  $\mathbf{S}^1 \times \mathbf{S}^1$ , as for any small stationary surface (Lawlor [16]), and  $S$  is stationary for prescribed  $\Omega$ -volume. It is the same story for any stationary product of spheres or of minimal submanifolds of spheres.

## 5. CALIBRATIONS

The classical theory of calibrations (see [[18], §6.4] and references therein) says that if there is a closed form  $\omega$  on a smooth Riemannian manifold such that  $|\omega| \leq 1$  with equality on the tangent planes to a surface  $S$ , then  $S$  is area minimizing in its homology class. The form  $\omega$  is called a calibration of  $S$ . Morgan [22] noted that for hypersurfaces if the condition that  $d\omega$  be 0 is relaxed to the condition that  $d\omega$  be a constant multiple of the volume form, then  $S$  still minimizes area for prescribed volume. In particular, constant-mean-curvature graphs have such “d-constant calibrations” [22], citing [[18], §6.1].

The following proposition is a trivial extension to prescribed  $\omega$ -volume in general codimension.

**Proposition 5.1.** *If  $\omega$  attains its maximum value (say 1) everywhere on a surface  $S$  in a smooth Riemannian manifold, then  $S$  is isoperimetric for prescribed  $\omega$ -volume.*

*Remarks.* In particular, every smooth surface is isoperimetric for some smooth  $\omega$ .

By Proposition 2.5, unless  $d\omega \lrcorner T$  vanishes on every unit tangent plane  $T$  to  $S$ , the mean curvature vector  $\mathbf{H}$  of  $S$  is proportional to  $d\omega \lrcorner T$ .

*Proof of Proposition 5.1.* If  $S'$  has the same  $\omega$ -volume, then

$$|S| = \int_S \omega = \int_{S'} \omega \leq |S'|.$$

*Remarks.* Consider a smooth exact  $(m+1)$ -form  $\Omega$  and a surface  $S$  with unit tangent planes  $T$  and mean curvature vector  $\mathbf{H}$  such that  $\mathbf{H} = \Omega \lrcorner T$ . By Proposition 2.5,  $S$  is stationary for given  $\Omega$ -volume. By Proposition 5.1,  $S$  is area minimizing for given  $\Omega$ -volume. This probably always holds locally *a la* Lawlor [16]. Conversely, if  $S$  is area minimizing for given  $\Omega$ -volume, there is probably in some generalized weak sense a calibration  $\omega$  *a la* Federer [13].

## REFERENCES

- [1] William K. Allard, *On the first variation of a varifold*, Ann. of Math. **95** (1972), 417-491.
- [2] William K. Allard, *On boundary regularity for Plateau's problem*, Bull. Amer. Math. Soc. **75** (1969), 522-523.
- [3] Fred Almgren, *Optimal isoperimetric inequalities*, Indiana Univ. Math. J. **35** (1986), 451-547.
- [4] Frederick J. Almgren, Jr.,  *$Q$ -valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two*, Bull. Amer. Math. Soc. **8** (1983), 327-328.
- [5] João L. Barbosa and Manfredo do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), 339-353.
- [6] Enrico Bombieri, Ennio De Giorgi, and Enrico Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243-268.
- [7] Ulrich Dierkes, Stefan Hildebrandt, Albrecht Kuster, and Ortwin Wohlrab, *Minimal Surfaces II: Boundary Regularity*, Springer-Verlag, NY, 1992.
- [8] Frank Duzaar and Martin Fuchs, *On the existence of integral currents with prescribed mean curvature vector*, Manuscripta Math., **67** (1990), 41-67.
- [9] Frank Duzaar and Martin Fuchs, *On integral currents with constant mean curvature*, Rend. Sem. Univ. Padova, **85** (1991), 79-103.
- [10] Frank Duzaar and Martin Fuchs, *A general existence theorem for integral currents with prescribed mean curvature form*, Boll. Un. Mat. Ital. B (7), 6-B (1992), 901-912.
- [11] Frank Duzaar and Klaus Steffen,  *$\lambda$  minimizing currents*, Manuscripta Math., **80** (1993), 403-407.
- [12] Klaus Ecker, *Area-minimizing integral currents with movable boundary parts of prescribed mass*, Ann. Inst. H. Poincaré, **6** (1989), 261-293.
- [13] Herbert Federer, *Real flat chains, cochains, and variational problems*, Indiana Univ. Math. J. **24** (1974/75), 351-407.
- [14] Robert Gulliver, *Existence of surfaces with prescribed mean curvature vector*, Math. Z., **131** (1973), pp. 117-140.
- [15] Robert Gulliver, *Necessary conditions for submanifolds and currents with prescribed mean curvature vector*, in Seminar on Minimal Submanifolds, Enrico Bombieri, ed., Ann. of Math. Studies **103**, Princeton Univ. Press, 1983, 225-242.
- [16] Gary Lawlor, *A sufficient condition for a cone to be area-minimizing*, Mem. Amer. Math. Soc. **91** (446), 1991.
- [17] H. Blaine Lawson, *Compact minimal surfaces in  $S^3$* , 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. **XV**, Berkeley, Calif., 1968), 275-282.
- [18] Frank Morgan, *Geometric Measure Theory: a Beginner's Guide*, Academic Press, 2009.
- [19] Frank Morgan, *An isoperimetric inequality for the thread problem*, Bull. Austral. Math. Soc. (1997), 489-495.
- [20] Frank Morgan, *Measures on spaces of surfaces*, Arch. Rat. Mech. Anal. **78** (1982), 335-359.
- [21] Frank Morgan, *Perimeter-minimizing curves and surfaces in  $\mathbf{R}^n$  enclosing prescribed multi-volume*, Asian J. Math. **4** (2000), 373-382.

- [22] Frank Morgan, *Strict calibrations*, Matemática Contemporânea **9** (1995), 139-152.
- [23] Johannes C. C. Nitsche, *Vorlesungen über Minimalflächen*, Springer-Verlag, New York, 1975; Translation: *Lectures on Minimal Surfaces*, Cambridge Univ. Press, New York, 1989.
- [24] Isabel M. C. Salavessa, *Stability of submanifolds with parallel mean curvature in calibrated manifolds*, Bull. Braz. Math. Soc. **41** (2010), 495-530.
- [25] Isabel M. C. Salavessa, *Stable 3-spheres in  $\mathbb{C}^3$* , J. Math. Research **4** (2) (2012), 34-44.
- [26] Richard Schoen, *Minimal submanifolds in higher codimension*, Matemática Contemporânea **30** (2006), 169-199, [http://www.mat.unb.br/~matcont/30\\_11.pdf](http://www.mat.unb.br/~matcont/30_11.pdf)
- [27] Henry C. Wente, *A note on the stability theorem of J. L. Barbosa and M. do Carmo for closed surfaces of constant mean curvature*, Pacific J. Math. **147** (1991), 375-379.
- [28] Brian White, *The space of m-dimensional surfaces that are stationary for a parametric elliptic functional*, Indiana Univ. Math. J. **36** (1987), 567-602.
- [29] Shing Tung Yau, *Submanifolds with constant mean curvature. I. II. Amer. J. Math.* **96** (1974), 346-366; *ibid* **97** (1975), 76-100; MR0370443 (51 #6670).

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